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AN APPROACH FOR WAVE VELOCITY MEASUREMENT IN SOLID CYLINDRICAL RODS SUBJECTED TO ELASTIC IMPACT

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Abstract—Certain cross-sectional resonances of a long, solid, cylindrical rod, excited by transverse, elastic impact loading, may be measured by an experimental technique. The values of these resonance frequencies can be predicted knowing the material characteristics of the rod, but it is of greater interest to inversely solve for the material characteristics of the tested material from the experimentally obtained frequency values. In the case of portland cement concrete testing specifically, the bulk shear wave velocity of the material is important to know but difficult to measure. In this paper, the governing resonance equation will be manipulated and inverted, ultimately resulting in an expression of bulk shear wave velocity in terms of the nth ordered resonance frequency, Poisson's ratio, and cross-sectional solid rod radius. The operation is not tractable when performed symbolically, however, because of the presence of Bessel functions; therefore, this novel inversion will be achieved through the approximation of Bessel functions within the resonance equation with 2nd order Taylor series, resulting in a quadratic equation in normalized resonance frequency Ω. The roots of the quadratic equation may then be solved explicitly, resulting in two symbolic expressions for Ω, one of which is selected as the appropriate approximation. Manipulation of the selected root expression results in the desired symbolic expression for bulk shear wave velocity. With numerical examples from the literature, it is demonstrated that use of the series provides good approximation of the roots of the original resonance equation across a significant span of coefficient values and allows for sufficient inverse calculation of bulk shear wave velocity based on experimental results. The symbolic form of the inverted expression for bulk shear wave velocity is given in the Appendix. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Experimental and analytical studies concerning the transient behavior of beam-like structures subjected to dynamic loading have been undertaken for decades [Goldsmith and Cunningham (1956), Jones (1964)]. In these early analytical studies, simplifying approximations, such as the stationary phase approximation, were used in order to enable the calculation of the dynamic behavior. A more comprehensive and physically-based approach for such responses was developed by Flax *et al.*, and expanded by others, where the transient response of long, solid, cylindrical rods immersed in water and subjected to normally and obliquely incident ultrasonic plane waves was extensively examined [Flax *et al.* (1981), Maze *et al.* (1985), Boa *et al.* (1990)]. In these studies, it was found that the transient response of the submerged structure involves the superposition of excited cross-sectional resonance modes of the structure. In a recent study, the transient response of traction-free (in vacuum) long, solid, cylindrical structures subjected to transverse elastic impact point loading was empirically demonstrated also to be comprised of the superposition of measurable cross-sectional resonance modes [Lin and Sansalone (1992)], and an exact model for the prediction of the frequency value of these specific resonance modes, based on elastic guided wave theory, was subsequently proposed [Popovics (1994)].

From this elastic guided wave model, the frequency value of these resonance modes may be shown to be a function of the bulk wave velocities—V_L and V_S—and the radial size—a—of the solid, elastic rod; thus, the excited resonant frequencies may be easily predicted for a given solid rod structure subjected to an elastic impact. However, there are no published approaches which permit the converse calculation: the determination of the rod parameters upon symbolic inversion and manipulation of the resonance frequency equation, resulting in an expression for V_L or V_S in terms of the nth ordered resonance frequency, Poisson's ratio ν, and a of the specimen. Through the inversion of the resonance equation, that task is approached in this paper. Only one of the coupled bulk wave velocities needs to be determined if ν is known since V_L may be calculated from V_S, and vice versa: for an elastic material, Poisson's ratio is related to the bulk wave velocities by [Fung (1965)]

$$\zeta = V_S/V_L = \sqrt{\frac{1-2\nu}{2(1-\nu)}}. \tag{1}$$

The motivation for this work stems from the fact that the determination of V_S for portland cement concrete within structures is of interest, yet that value is difficult to measure with existing nondestructive test techniques [Swami (1971)].

REVIEW OF THEORY

The elastic guided wave based model for the specific resonance frequencies excited by transverse, elastic impact upon a solid cylindrical rod will now be reviewed. The derivations of elastic wave propagation in the isotropic, traction-free, cylindrical, solid rod are well known [Meeker and Meitzler (1964), Smith (1971), Zemanek (1972), Auld (1990)]. In brief, harmonic wave propagation along the infinite waveguide is assumed and the dynamic equations of motion and traction-free boundary conditions are satisfied in order to arrive upon the frequency dispersion relations; these equations relate harmonic frequency (ω) and propagating wavenumber (γ) for a specific waveguide. The equation describing the so-called cutoff frequencies is then obtained by setting $\gamma = 0$, wherein ω represents the cutoff frequency in radians. A true cutoff frequency is that which separates propagating from evanescent modes. Supplementary roots of the cutoff frequency equation exist when complex values of γ are admitted [Zemanek (1972)]. For the purpose of this work however, the cutoff frequency roots based only on real values of γ will suffice.

It has been demonstrated that the lowest valued, non-trivial, plane–strain cutoff modes, for wave modes of circumferential order $n = 2$ and greater, of a solid, long, cylindrical rod are the resonance modes excited by a transverse, elastic impact source [Popovics (1994)]. The integral value of circumferential order n describes the θ dependence of the out-of-plane displacements, u_r, of the modes as $u_r(r, \theta) = F_1(r) \cos(n\theta)$, where $F_1(r)$ is also a function of n, the bulk wave velocities, and the cutoff frequency. Figure 1 shows the geometry of the solid

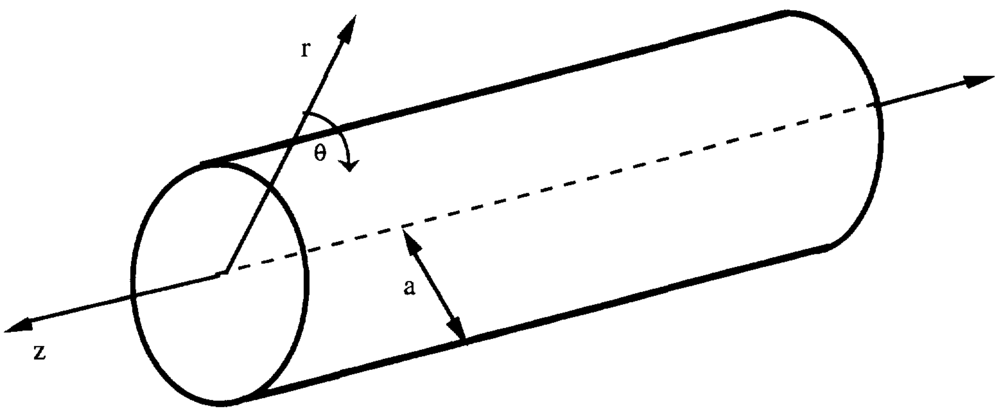

Fig. 1. Geometry of the infinite solid cylindrical rod and coordinate system used.

cylindrical rod structure as well as the coordinate system used. Following the convention of Gazis (1958), plane–strain modes are those which are comprised of coupled u_r and u_θ components. In the classic derivation it can be shown that the frequencies of these plane–strain modes are a function of V_L, V_S, and a of the solid rod being tested: the general cutoff frequency equation of plane–strain modes for a circular cross-sectioned, solid rod of radius a is given by

$$F_2(n,\zeta,\Omega) = \{2(n^2-1)[\Omega J_{n-1}(\Omega)-nJ_n(\Omega)]-\Omega^2 J_n(\Omega)\}[\zeta\Omega J_{n-1}(\zeta\Omega)-(n+1)J_n(\zeta\Omega)]$$
$$-J_n(\zeta\Omega)(n^2-1-(\Omega^2/2))[(2n^2+2n-\Omega^2)J_n(\Omega)-2\Omega J_{n-1}(\Omega)] = 0, \quad (2)$$

where J_n is an nth order Bessel function of the first kind, $\zeta = V_S/V_L = \sqrt{(1-2\nu)/2(1-\nu)}$, and $\Omega = \omega a/V_S$ is the normalized cutoff frequency where ω is the actual cutoff frequency in radians [Zenanek (1972)].

APPROACH FOR INVERSION

The resonance frequency values of interest are given by specific roots of the cutoff frequency equation (eqn (2)). As can be seen, this equation is comprised of nth and $(n-1)$th ordered Bessel functions of the first kind with arguments that are a function of bulk velocity ratio and normalized resonance frequency values: $J_n(\zeta,\Omega)$ and $J_{n-1}(\zeta,\Omega)$. Only the lowest valued, non-trivial frequency roots of eqn (2), for $n \geqslant 2$, are of interest. It is thus desirable to find an expression which relates the normalized resonance frequency value Ω to the other variables ζ, n for this special root of eqn (2). However, it is not possible to symbolically invert equations of the form of eqn (2), as desired, since the inverse of the Bessel function cannot be expressed in terms of known functions. An approach which makes this needed calculation possible is either to approximate eqn (2) through variational principles or to replace the $J_n(\zeta,\Omega)$ and $J_{n-1}(\zeta,\Omega)$ terms in eqn (2) with polynomial approximations of the respective functions. In either case, it is desirable to minimize computational complexity, since the approximation should be inverted in symbolic form, yet simultaneously maximize the sufficiency of the approximation.

Considerable work has been done on the use of variational principles for the approximation of the governing equations of the free vibrations of circular cylinders. This approach has been proven to be very successful for the estimation of modal frequencies and associated modal displacement patterns in cylindrical structures; only the numerical solution to an algebraic eigenvalue problem is ultimately required [Nelson *et al.* (1971)]. However, this approach has limited viability when the eigenvalues are to be solved in symbolic form. The Rayleigh quotient offers a method by which the estimation of an eigenvalue in symbolic form is possible, but this eigenvalue is generally that which is greatest in absolute value of the system [Kreyszig (1988)]. In this work, only the lowest valued root of eqn (2) is of interest. Thus, variational principles, as found in the literature, are not applicable for this work.

Considering the latter approach, any polynomial approximation of the Bessel functions should be valid for orders 1–7 and for Bessel function argument values in the range of 1–8 in this case. Accurate polynomial approximations of Bessel functions have been tabulated [Olver (1972)], but these approximations are not valid in the desired argument range: specific approximations are given either for the range $0 \leqslant x \leqslant 3$ or for the range $3 \leqslant x \leqslant \infty$, where x is the Bessel function argument. Bessel function approximations which make use of Chebyshev series are valid within the desired argument range of $0 \leqslant x \leqslant 8$, but expressions for only the zeroth and first order functions could be found [Clenshaw and Picken (1966)]. Thus, power series appear to be the only polynomial approximations which are viable for Bessel function approximation and meet the order and argument range requirements. Note that power series have been used successfully to approximate Bessel functions within guided wave frequency equations when the cross-sectional radius of the solid rod is known to be small [Love (1944)].

The general power series approximation of the Bessel function of the first kind of order n near $x = 0$ is [Hildebrand (1976)]

$$\tilde{J}_n(x) = \sum_{k=0}^{N} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}, \tag{3}$$

where the order of the series is defined as $2N+n$. Alternatively, the Taylor series approximation of the Bessel function of the first kind of order n near $x = x_o$, where x_o is arbitrary and the Bessel function is assumed to be regular at $x = x_o$, is given as [Hildebrand (1976) p. 122]

$$\tilde{J}_n(x) = \sum_{k=0}^{N} \frac{J_n^k(x_o)}{k!}(x-x_o)^k. \tag{4}$$

The raised (k) in eqn (4) represents differentiation of order k with respect to x; the order of the series is defined as N. The derivative of the Bessel function can be expressed as

$$\mathrm{d}/\mathrm{d}x J_n(\eta x) = \eta/2[J_{n-1}(\eta x) - J_{n+1}(\eta x)] \tag{5}$$

where η is a constant [Hildebrand (1976) p. 149]. When $x_o = 0$, eqns (4) and (5) reduce to eqn (3). Either eqn (3) or eqns (4) and (5) may be used to represent the Bessel functions within eqn (2). Once the Bessel functions within eqn (2) are replaced by such polynomial expressions, the determination of the needed inverse relations becomes tractable, since symbolic solutions of polynomial equations of fourth order and below are well known. Thus, the Bessel functions in eqn (2) were replaced with both types of series approximations and evaluated; the symbolic mathematics manipulation package Mathematica$_R$ was used for all calculations. Both series types were expanded with respect to Ω; in either case, the approximation of eqn (2) may be written in the form

$$\sum_{k=0}^{N} F_2^{(k)}(n, \zeta, \Omega_o)\frac{(\Omega-\Omega_o)^k}{k!} = 0 \tag{6}$$

where the raised (k) now represents differentiation of order k with respect to Ω, Ω_o is an initial estimate of the desired root, and the order of the series is defined as N. The problem of approximating a specific root of eqn (2) thus reduces to the concurrence of the determination of an appropriate series order, the choice of Ω_o, and the selection of a root from the N possible solutions of eqn (6). The series parameters Ω_o and N must therefore be selected so to strike a balance between sufficient approximation and computational simplicity, and the selected value of Ω_o should be generally close to the expected root. However, the use of nontrivial values of Ω_o leads to significant algebraic complication because of the involvement of Bessel functions and derivatives of Bessel functions evaluated at Ω_o—see eqns (4) and (5)—within the series coefficients. Thus, two prospective values of Ω_o were chosen in this paper: $\Omega_o = 0$ and $\Omega_o = 0.455\pi n/2\zeta$. The second Ω_o value—$0.455\pi n/2\zeta$—was chosen as the point to be expanded about since this expression serves as a rough approximation of the nth order resonance frequency excited in a solid rod when $\zeta = 0.6125$. This expression for Ω_o is a manipulated form of the expression, valid for $v = 0.20$, $f_o = 0.455 V_L n/4a$ where f_o is the predicted cutoff frequency in Hertz [Lin and Sansalone (1992)]. It should be noted that the validity of the use of f_o itself is highly dependent upon the value of material v; failure to account for reasonable variations in material v may lead to calculated f_o values which are over 10% in error [Popovics (1994)]. Thus, the use of Ω_o itself for the inverse calculation of V_S is not sufficient.

Table 1. Exact and series approximations of Ω root values of eqn (2) for various circumferential orders n. $\zeta = 0.6125$

		Series approx. about $\Omega_o = 0$			Series approx. about $\Omega_o = 0.455\pi n/2\zeta$		
n	Exact	$N = 5$	$N = 10$	$N = 15$	$N = 2$	$N = 3$	$N = 4$
2	2.34444	2.37803	2.34450	2.34444	2.34444	2.34444	2.34444
3	3.59456	‡	3.59584	3.59456	3.59512	3.59455	3.59456
4	4.67504	‡	4.68887	4.67506	4.67504	4.67504	4.67504
5	5.68733	‡	5.84939	5.68746	5.68515	5.68733	5.68735
6	6.66663	‡	‡	6.66714	6.62529	6.66654	6.66702
7	7.62750	‡	‡	7.62838	‡	7.62798	7.63158

‡ Complex valued roots.

Ultimately, the appropriate order of the series is that minimum value which provides sufficient approximation and also results in an expression which can be inverted in symbolic form. The sufficiency of the various series approximations was tested by comparing the lowest valued, non-trivial root of eqn (2) with a specific selected root from the approximation expression (expression (6)) for circumferential orders $n = 2$–7; the selected approximation root, of the N possible roots, was that which was purely real and clearly closest in value to the exact root for $\zeta = 0.6125$. In all cases, the appropriate approximation root was determined easily. The so-called exact roots of eqn (2) were actually obtained numerically: Newton's method of root determination was used with 6-digit precision upon the substitution of ζ and n values in eqn (2). The same method was used to obtain roots of eqn (6). Since Newton's method was utilized, the direction of root convergence cannot be predicted. The word "exact" was used to distinguish the actual roots of eqn (2) from the roots of eqn (6), the series approximation of eqn (2). The results of these trials are presented in Table 1. In this table, the root values of eqn (2) are compared to the selected roots of the approximations, utilizing a series expansion about $\Omega_o = 0$ (eqn (3)) and about $\Omega_o = 0.455\pi n/2\zeta$ (eqns (4) and (5) respectively), for a variety of series orders.

Firstly, it is clear from the data in Table 1 that all of the approximations worsen as the circumferential order n increases. Considering the series about $\Omega_o = 0$, the data in Table 1 show that a 15th order series is needed to approximate the exact normalized frequency root values of eqn (2) within ± 0.001 for n orders up to 7. In fact, the 5th and 10th order power series cannot supply real valued roots for the higher n orders. Although the form of the series approximation of eqn (2) with $\Omega_o = 0$ is relatively simple and the 15th order approximation of eqn (2) is quite good, this substitution results in the formation of a 15th order polynomial in the Ω term (eqn (6)) which eventually needs to be solved. Of course, polynomials of order higher than 4 cannot be solved symbolically, so the use of the 15th order power series is not feasible for this purpose. It can be seen, however, that the use of the series about $\Omega_o = 0.455\pi n/2\zeta$ results in sufficient approximations requiring much lower series orders. As seen in Table 1, the 2nd, 3rd, and 4th order series supply good approximations of eqn (2); specifically, the 3rd order series about $\Omega_o = 0.455\pi n/2\zeta$ approximates the exact Ω root values within ± 0.0006 for n orders 2–7, while the 2nd order series supplies the same degree of approximation for n orders 2–4. Note that the use of the 2nd order series results in a quadratic equation in the Ω term as an approximation of eqn (2). The symbolic solution of quadratic equations is significantly more tractable than that of cubic or quartic equations; thus, the 2nd order series expansion about $\Omega_o = 0.455\pi n/2\zeta$ was selected as most appropriate for the purpose of obtaining approximate symbolic expressions of eqn (2) root values for n orders 2–5. If approximations of the higher circumferential order modes ($n = 6, 7, \ldots$) are needed, the 3rd order series approximation may be used, resulting in a cubic equation in the Ω term. Once the 2nd order series was substituted into eqn (2) and a quadratic equation in the Ω term was obtained, this new equation was then solved symbolically resulting in two Ω roots in terms of ζ and n. Of the two obtained roots, the one which supplied values clearly closest to those of the exact form of eqn (2), for $\zeta = 0.6125$ and $n = 2$–5, was selected as the appropriate root. This ultimately results in

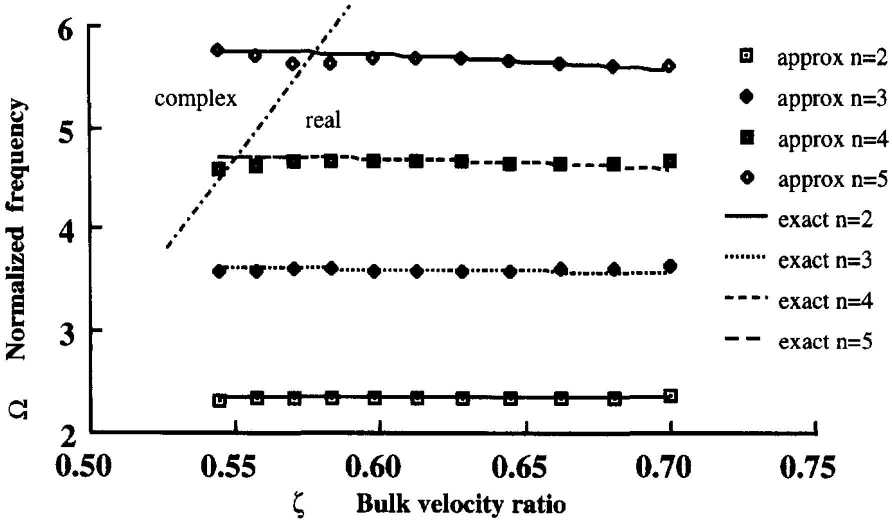

Fig. 2. Comparison of exact and 2nd order series approximation of normalized frequency root values Ω of eqn (2) as a function of bulk wave velocity ratio ζ for circumferential orders $n = 2$–5.

an expression of normalized resonance frequency in terms of bulk velocity ratio and circumferential order:

$$\Omega = F_3(\zeta, n), \tag{7}$$

where $F_3(\zeta, n)$ is a function of ζ and n only.

The drawback of using the series approximation about $\Omega_o = 0.455\pi n/2\zeta$, as seen upon inspection of eqn (6), is the arithmetic complexity of series coefficients containing multiple derivatives of Bessel functions evaluated at $\Omega_o = 0.455\pi n/2\zeta$. A manipulated form of eqn (7), for $n = 2$, is presented in the Appendix where the arithmetic complexity is readily seen. Despite this complexity, the quadratic approximation of the desired roots of eqn (2) appears to work quite well and enables behavioral study through parametric variation. Figure 2 shows the excellent agreement between the exact and approximate root values of eqn (2), for $n = 2$–5, across a span of ζ values. The "exact" values of eqn (2) were numerically calculated, as described earlier, whereas the approximate values were obtained by simply evaluating eqn (7) upon substitution of specific values of ζ and n. It should be noted that the nature of root approximation (eqn (7)) changes from purely real to complex in the region of higher n values and lower ζ values (below $\zeta = 0.54$ for $n = 4$ and below $\zeta = 0.58$ for $n = 5$), as shown in Fig. 2. When complex values of eqn (7) were encountered, only the real portion was utilized. Although complex values of eqn (7) were considered undesirable, Fig. 2 shows that the real portion of these values approximate the exact roots of eqn (2) fairly well.

Upon simple substitution, the root approximation (eqn (7)) may be expressed as the resonance frequency (nth order) in terms of the bulk wave velocities, solid rod radius, and circumferential order:

$$f = F_4(V_L, V_S, a, n) \tag{8}$$

where $f = \omega/2\pi$. The limits of validity of eqn (8) are shown in Fig. 3 for n orders ranging from 2–5. Specifically, Fig. 3 shows the excellent agreement between the exact root values (eqn (2)) and approximate root values (eqn (8)), for $n = 2$–5, across a span of V_S values. It is clear, however, that the root approximation worsens as the mode order n increases and V_S values tend away from the median value shown. Again, the "exact" values in Fig. 3 were numerically calculated as described earlier whereas the approximate values were obtained by simply evaluating eqn (8) upon the substitution of specific values of V_L, V_S, n, and a. Note that the range of values of bulk wave velocity were chosen so to comprehensively

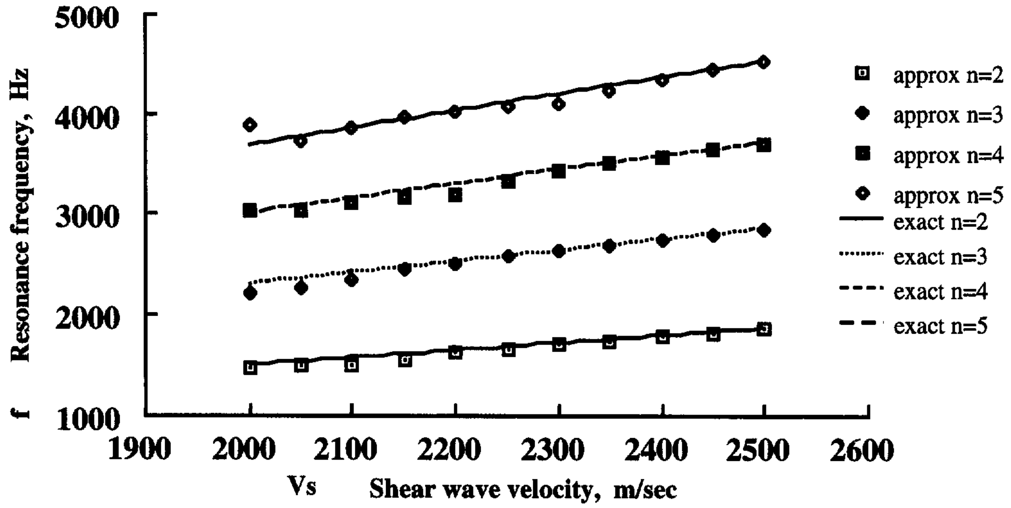

Fig. 3. Comparison of exact and 2nd order series approximation of resonance frequency values as a function of shear wave velocity for mode orders $n = 2$–5. $V_L = 4000$ m/sec (13124 ft/sec). $a = 0.5$ m (19.7 inch).

represent the material portland cement concrete. Figure 4 demonstrates the excellent agreement between the exact root values (eqn (2)) and approximate root values (eqn (8)) for a wide range of solid rod radii and for all mode orders n selected.

Note that analogous approach for the symbolic inversion of resonance frequency equations for hollow rods may also be developed. The extension of the solid rod formulation to the hollow rod case is expected to be tractable, except that Bessel functions of the first *and* second kind must be considered, and a new, appropriate point about which to expand the Taylor series must be determined.

NUMERICAL APPLICATION

Now that the validity of using a 2nd order Taylor series to approximate roots of eqn (2) is established, we can algebraically manipulate eqn (8) to give an expression for V_S in terms of V_L, a, n, and the associated resonance frequency value. It has been demonstrated that changing V_S with constant V_L has roughly one-hundred-fold more of an effect on the

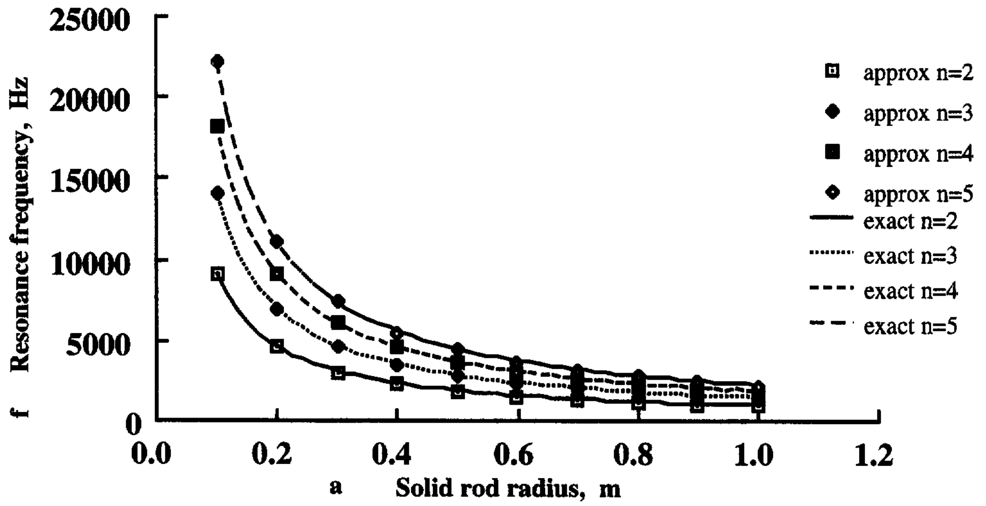

Fig. 4. Comparison of exact and 2nd order series approximation of resonance frequency values as a function of solid rod radius for mode orders $n = 2$–5. $V_L = 4000$ m/sec (13124 ft/sec). $V_S = 2450$ m/sec (8038 ft/sec). $\zeta = 0.6125$.

resonance frequency values (eqn (2)) than changing V_L with constant V_S does [Popovics (1994)]. Considering this, V_S should be solved for rather than V_L, with V_L terms replaced by an expression comprised of V_S and ν following eqn (1); this approach has the effect of minimizing any error propagation in the calculation of bulk wave velocity. This manipulation results in an expression for resonance frequency which is a function of V_S, a, and ν. It is found that eqn (8) then takes the form

$$f = \frac{\sqrt{2}F_5(n,\nu)V_S}{2a} \tag{9}$$

where $F_5(n,\nu)$ is a function of n and ν only. Equation (9) may thus be further manipulated to give the desired expression for shear wave velocity:

$$V_S = \frac{\sqrt{2}af}{F_5(n,\nu)}. \tag{10}$$

Because of arithmetic complexity, the complete form of eqn (10) is presented in the Appendix only for $n = 2$. Note that the mathematical form of eqn (10) suggests that V_S is directly related to both a and the lowest valued resonance frequency for each n, and that f and a are inversely related. Following from the parametric study presented as Figs 2–4, resonance frequency values of circumferential mode orders $n = 2$, $n = 3$, and $n = 4$ could be used with confidence in conjunction with eqn (10); that is, these modes and associated frequency values could be used in order to obtain redundant estimates of V_S, enabling significant error reduction and improved confidence in the results, However, ν and a of the rod must be known. As an aside, note that ν may also be nondestructively determined by an approach which is based on the measurement of resonances of the solid rod [Popovics (1994)]. This approach for ν determination will be detailed in a subsequent communication.

An example of redundant V_S calculation, based on experimentally determined resonance frequency data from the literature, for a solid portland cement concrete rod with known a and ν, is shown now. The elastic impact generated resonance frequencies of a solid rod specimen for $n = 2$–5 are obtained by an experimental technique with a frequency resolution of 488 Hz. These data are presented in Table 2. Also shown are redundant V_S estimates, calculated using eqn (10), using the obtained resonance frequencies and knowing a and ν of the test material; eqn (10) was numerically evaluated as before. For the sake of brevity, the full forms of eqn (10) for n values 3, 4, and 5 are not presented. However, the value of the $F_5(n,\nu)$ term used for the calculation of V_S with eqn (10), as well as the calculated velocity values themselves, are also presented in Table 2 for each n value. Note that the uncertainty of the measured resonance frequencies, due to insufficient experimental frequency resolution of the data from the literature, results in an uncertainty of calculated V_S. Nevertheless, the calculated values of V_S, using eqn (10), compare very favorably with the actual values for $n = 2$–4: the actual V_S values lie within the range of calculated values for each of these circumferential orders. For the case of $n = 5$, however, the actual V_S value lies outside of the range of the calculated estimate. The redundancy of V_S estimates proves to be useful: the average of the calculated values for $n = 2$–4 is 2411.68 m/sec which is

Table 2. Comparison of calculated redundant V_S values (eqn (10)) from experimentally obtained, n order resonance frequencies with actual values. $a = 0.20$ m (7.9 inch), $\nu = 0.20$

n	Measured resonance frequency (Hz)*	$F_5(n,\nu)$	Calculated V_S (m/sec)	Actual V_S (m/sec)*
2	4400 ± 244	0.527686	2358.43 ± 130.8	2449.5
3	6800 ± 244	0.809194	2376.85 ± 85.3	2449.5
4	9300 ± 244	1.05227	2499.77 ± 65.6	2449.5
5	10700 ± 244	1.27963	2365.07 ± 53.9	2449.5

* Data taken from Lin and Sansalone (1992), p. 890.

closer to the actual value of 2449.5 m/sec than any individual calculated estimate. Thus, the redundancy of calculated estimates acts to overcome the effect of insufficient experimental frequency resolution.

CONCLUSIONS

The resonances in an isotropic solid rod excited by elastic impact can be modeled as cutoff frequencies in an elastic guided wave approach; the governing equation may be approximated through the use of 2nd order series which substitute for Bessel functions within the equation. The approximation is shown to be very good for circumferential orders $n = 2$–5, across a significant span of coefficient values. This novel approximation allows for symbolic manipulation of the resonance equation and explicit, symbolic solution for V_S in terms of the nth ordered resonance frequency, v, and a. This relation can be used in conjunction with experimental data for nondestructive measurement of V_S in cylindrical, solid structures, including those comprised of portland cement concrete, subjected to elastic impact, as shown. Specifically, measured resonance frequency values of circumferential mode orders $n = 2$, $n = 3$, and $n = 4$ may be used with the 2nd order Taylor series approximation to obtain redundant estimates of V_S, enabling significant error reduction and improved confidence in the results. Caution should be used when estimating V_S in this manner with the $n = 5$ mode, especially when $\zeta < 0.58$, since the approximation worsens as the circumferential order n of the modes increases and the value of ζ decreases. Extension of this approach to hollow rod structures is tractable.

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APPENDIX

The full form of the inverted, approximated resonance frequency equation, based on 2nd order series substitution about $\Omega_o = 0.455\pi n/2\zeta$, is presented for circumferential order $n = 2$. The expression gives the bulk shear wave velocity—V_S—in terms of the solid rod radius—a— Poisson's ratio—ν— and the nth ordered, lowest valued, plane–strain cutoff frequency—f. In the expressions, J_n represents the nth order Bessel function of the first kind. For all values of n, the obtained expression has the form

$$V_S = (\sqrt{2}af)\left(\frac{\Phi_3\Phi_4}{-\Phi_1-\sqrt{(\Phi_1)^2-4\Phi_2\Phi_3}}\right). \tag{A1}$$

The forms of the Φ_1, Φ_2, and Φ_3 terms change for each value of n; these expressions and Φ_4 are presented below for $n = 2$.

$$\begin{aligned}
\Phi_1 = {} & \frac{-5.509511732757821\,10^{613}\,J_0\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^2} + \frac{6.987887687125559\,10^{613}\,J_0\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^4} \\
& + \frac{5.124724359143601\,10^{613}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^5} - \frac{4.198849053568637\,10^{612}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^3} \\
& + \frac{5.670417460789501\,10^{613}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4} + 9.825538444929579\,10^{613}\,J_2\left(\frac{2.02151}{\Phi_4}\right) \\
& - \frac{5.387859298073472\,10^{614}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^2} + \frac{3.187648195218217\,10^{614}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^4} \\
& - \frac{3.918344975717834\,10^{613}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^6} - \frac{9.001372616254572\,10^{613}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^5} \\
& + \frac{2.166545059260146\,10^{614}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^3} - \frac{4.458883498614269\,10^{613}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4},
\end{aligned}$$

$$\begin{aligned}
\Phi_2 = {} & \frac{3.63453444399928\,10^{612}\,J_0\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^2} - \frac{6.85803171976813\,10^{612}\,J_0\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^4} \\
& - \frac{4.72695211040991\,10^{612}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^5} - \frac{1.177866719985518\,10^{612}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^3} \\
& - \frac{1.172526059569987\,10^{612}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4} - 1.437935316661618\,9\,10^{613}\,J_2\left(\frac{2.02151}{\Phi_4}\right) \\
& + \frac{5.618882909355802\,10^{613}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^2} - \frac{3.337582762289173\,10^{613}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^4} \\
& + \frac{4.457117409879037\,10^{612}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^6} + \frac{9.136639502873643\,10^{612}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^5}
\end{aligned}$$

$$-\frac{2.01359799750291\,10^{613}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^3}-\frac{7.012621253457394\,10^{611}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4},$$

$$\Phi_3=\frac{1.7195227142663461\,10^{614}\,J_0\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^2}-\frac{1.746535494446084\,10^{614}\,J_0\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^4}$$

$$-\frac{1.3393156974552397\,10^{614}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^5}-\frac{4.842545029056418\,10^{611}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^3}$$

$$-\frac{1.2594867548910211\,10^{614}\,J_1\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4}-3.747829513087562\,10^{614}\,J_2\left(\frac{2.02151}{\Phi_4}\right)$$

$$+\frac{1.4550118511542269\,10^{615}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^2}-\frac{7.984455957878915\,10^{614}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^4}$$

$$+\frac{8.611747199379854\,10^{613}\,J_2\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^6}+\frac{2.191326303413696\,10^{614}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^5}$$

$$-\frac{5.63273423463625\,10^{614}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4^3}+\frac{2.095442016422345\,10^{614}\,J_3\left(\frac{2.02151}{\Phi_4}\right)}{\Phi_4},$$

and

$$\Phi_4=\sqrt{\frac{1-2\nu}{1-\nu}}=\sqrt{2}\zeta.$$